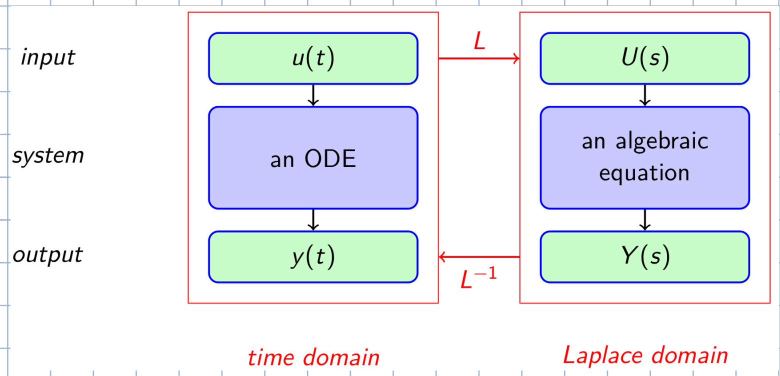
Laplace transforms are another way to transfer a function into a different domain.
It extends idea of Fourier transforms and enables us to apply it to a much wider range of signals and systems.



Widely used in engineering because:

- they work for many more signals & systems than Fourier.
- systematic method for solving linear ODEs
- provide general way to formulate a transfer function of an input-output system

Laplace transforms are in the "s-domain" or "Laplace domain"

They have a variable s that corresponds to characteristic decay rates, in the same way ω in the Fourier transform corresponds to characteristic freqs.

$$\boxed{\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\, dt}$$

$$F(s) = \mathcal{L}[f(t)] \quad \longleftrightarrow \quad \mathcal{L}^{-1}[F(s)] = f(t)$$

Comparing Laplace & Fourier:

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

$$\mathcal{F}[f(t)] = \int_{-\infty}^\infty e^{-j\omega t} f(t)\, dt$$

- Laplace transform does not include complex aspect. Uses decaying exponential e^{-st} rather than oscillating $e^{-j\omega t}$

  ↳ Laplace can deal with signals that don't have finite energy.

- Lower limit for Laplace is 0 instead of $-\infty$
  - ↳ Laplace used for processes that occur forward in time

Laplace transform is linear

Laplace Transforms of Common Functions :

$$\mathcal{L}[1] = \frac{1}{s}$$

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}[e^{-at}] = \frac{1}{s+a}$$

$$\mathcal{L}[\cos(\omega t)] = \frac{s}{s^2+\omega^2}$$

$$\mathcal{L}[\sin(\omega t)] = \frac{\omega}{s^2+\omega^2}$$

REFER TO TABLE FOR MORE

Derivatives of Laplace Transforms :

$$\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n F}{ds^n}$$

e.g $\mathcal{L}[t f(t)] = -\frac{dF}{ds}$

Solving ODEs with Laplace Transform :

1. Take Laplace transform of ODE e.g $x(t)$
2. Solve for Laplace transform of output e.g $X(s)$
3. Simplify until you have recognisable parts
4. Use knowledge, memory & Laplace transform tables to invert transform
5. We now have solution in time-domain $x(t)$

Laplace Transform of Derivatives :

$$\mathcal{L}\left[\frac{d^n f}{dt^n}\right] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - f^{(n-1)}(0)$$

e.g $\mathcal{L}\left[\frac{d^2 f}{dt^2}\right] = s^2 F(s) - s f(0) - f'(0)$

derivative of time-domain function evaluated at $t=0$ → constant

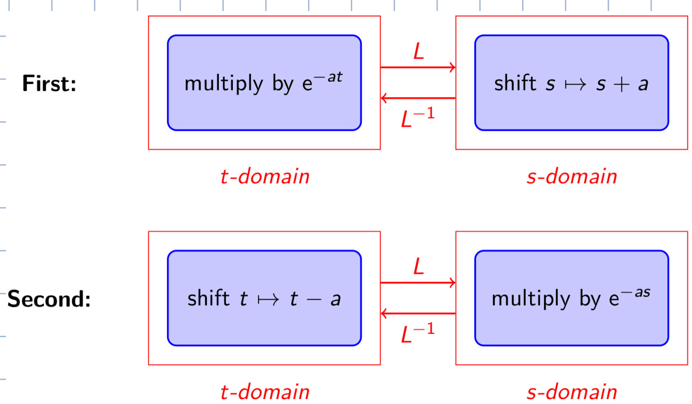
Shifting Theorems :

1st : A shift in s corresponds to multiplication by an exponential in t

$$\mathcal{L}[e^{-at} f(t)] = F(s+a)$$

2nd : A shift in t corresponds to multiplication of exponential in s

$$\mathcal{L}[H(t-a) f(t-a)] = e^{-as} F(s)$$

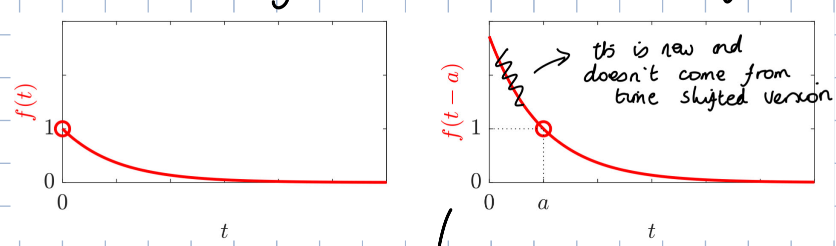


Heaviside Step Function : a mathematical switch.

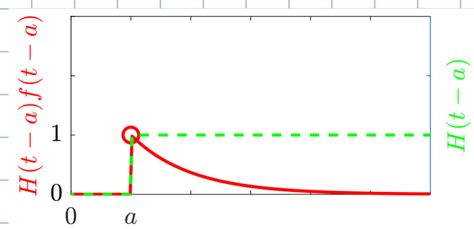
$$H(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

→ ensures we don't invent new signal when we make shift in t

e.g if $f(t) = e^{-t}$



we want



Transfer Functions :

Laplace transforms can be used to analyse input - output systems, exactly as we did for the Fourier transform.

e.g we can consider a linear time - domain system with output $y(t)$ and input $u(t)$.

taking a Laplace transform we can get an equation

$$Y(s) = G(s)\, U(s)$$

where $G(s)$ is the transfer function

Definition : for an autonomous (time - invariant) linear system the transfer function $G(s)$ is the ratio $Y(s)/U(s)$ of the Laplace transform of the output to the Laplace transform of the input.

In general the transfer function of a system can be written in the form :

$$G(s) = \frac{Q(s)}{P(s)}$$

where $P(s)$ & $Q(s)$ are polynomial functions in s.

A system is said to be asymptotically stable if the roots of $P(s)$ (called the poles of the transfer function) are in the left half of the complex plane.

-ve real part